Artham Resource Material

For Revised Syllabus Session 2024-25

EXEMPLAR SOLUTIONS

MATHS

Chapter 5 : Continuity and Differentiability

Class 12

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Chapter 5:
Continuity and Differentiability

Exercise 5.3

Short Answer (S.A.)

1. Examine the continuity of the function f (x) = $x^3 + 2x^2 - 1$ at x = 1

Solution:

We know that, y = f(x) will be continuous at x = a if,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$$

Given: $f(x) = x^3 + 2x^2 - 1$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0}(1+h)^3 + 2(1+h)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \to 1} f(x) = (1)^3 + 2(1)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{\to}(1+h)^3 + 2(1+h)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = 2.$$

Thus, f(x) is continuous at x = 1.

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x) = \begin{cases} 3x+5, \text{ if } x \geq 2 \\ x^2, \text{ if } x < 2 \end{cases}$ **at x = 2**

Solution:

Checking the continuity of the given function, we have

$$\lim_{x \to 2^+} f(x) = 3x + 5$$
$$= \lim_{h \to 0} 3(2+h) + 5 = 11$$

$$\lim_{x \to 2} f(x) = 3x + 5 = 3(2) + 5 = 11$$

$$\lim_{x \to 2^-} f(x) = x^2 = \lim_{h \to 0}(2-h)^2$$
$$= \lim_{h \to 0}(2)^2 + h^2 - 4h = (2)^2 = 4$$

Now, since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) \neq \lim_{x \to 2} f(x)$

Thus, f(x) is discontinuous at x = 2.

3. $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & if\ x \neq 0 \\ 5, & if\ x = 0 \end{cases}$ **at x = 0**

Solution:

Checking the right hand and left hand limits of the given function, we have

$$\lim_{x \to 0^-} f(x) = \frac{1 - \cos 2x}{x^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2(0 - h)}{(0 - h)^2} = \lim_{h \to 0} \frac{1 - \cos(-2h)}{h^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2h}{h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 h}{h^2} \qquad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h \to 0} \frac{2 \sin h}{h} \cdot \frac{\sin h}{h} = 2.1.1 = 2 \qquad \left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\lim_{x \to 0^+} f(x) = \frac{1 - \cos 2x}{x^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2(0 + h)}{(0 + h)^2} = \lim_{h \to 0} \frac{1 - \cos 2h}{h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 h}{h^2} = \frac{2 \sin h}{h} \cdot \frac{\sin h}{h} = 2.1.1 = 2$$

$$\lim_{x \to 0} f(x) = 5$$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0} f(x)$

Therefore, the given function f(x) is discontinuous at x = 0.

4. $f(x) = \begin{cases} \dfrac{2x^2 - 3x - 2}{x - 2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ **at x = 2**

Solution:

The given fucntion at x ≠ 0 can be rewritten as,

$$f(x) = \frac{2x^2 - 3x - 2}{x - 2}$$

$$= \frac{2x^2 - 4x + x - 2}{x - 2} = \frac{2x(x - 2) + 1(x - 2)}{x - 2}$$

$$= \frac{(2x + 1)(x - 2)}{x - 2} = 2x + 1$$

Now,

$$\lim_{x \to 2^-} f(x) = 2x + 1$$

$$= \lim_{h \to 0} 2(2 - h) + 1 = 4 + 1 = 5$$

$$\lim_{x \to 2^+} f(x) = 2x + 1$$

$$= \lim_{h \to 0} 2(2 + h) + 1 = 4 + 1 = 5$$

$$\lim_{x \to 2} f(x) = 5$$

$$\text{As } \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 5$$

Thus, f(x) is continuous at x = 2.

5. $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x=4 \end{cases}$ **at x = 4**

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to 4^-} f(x) = \frac{|x-4|}{2(x-4)} \qquad \begin{bmatrix} \text{for } x < 4, |x-4| = -(x-4) \\ \text{for } x > 4, |x-4| = (x-4) \end{bmatrix}$$

$$= \lim_{h \to 0} \frac{-[4-h-4]}{2[4-h-4]} = \lim_{h \to 0} \frac{h}{-2h} = -\frac{1}{2}$$

$$\lim_{x \to 4^+} f(x) = \frac{|x-4|}{2(x-4)} = \lim_{h \to 0} \frac{[4+h-4]}{2[4+h-4]} = \frac{1}{2}$$

$$\lim_{x \to 4} f(x) = 0$$

$$\therefore \quad \lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x) \neq \lim_{x \to 4} f(x)$$

Thus, f(x) is discontinuous at x = 4.

6. $f(x)=\begin{cases} |x|\cos\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$ **at x = 0**

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to 0^-} f(x) = |x| \cos\frac{1}{x}$$

$$= \lim_{h \to 0} |0-h| \cos\frac{1}{(0-h)} = \lim_{h \to 0} h \cos\frac{1}{h}$$

$$= 0 \qquad \left[\because \cos\frac{1}{x} \text{ oscillate between } -1 \text{ and } 1\right]$$

$$\lim_{x \to 0^+} f(x) = |x| \cos\frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)} = \lim_{h \to 0} h.\cos \frac{1}{h} = 0$$

$$\lim_{x \to 0^+} f(x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0$$

Thus, the given function f(x) is continuous at x = 0.

7. $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & if\ x \neq 0 \\ 0, & if\ x = a \end{cases}$ **at x = a**

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to a^-} f(x) = |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a-h-a|.\sin \frac{1}{a-h-a} = \lim_{h \to 0} h.\sin \frac{1}{-h}$$

$$= \lim_{h \to 0} -h.\sin \frac{1}{h} \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= 0 \times [\text{a number oscillating between} -1 \text{ and } 1]$$

$$= 0$$

$$\lim_{x \to a^+} f(x) = |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a+h-a|.\sin \frac{1}{a+h-a} = \lim_{h \to 0} h.\sin \frac{1}{h}$$

$$= 0 \times [\text{a number oscillating between} -1 \text{ and } 1]$$

$$\lim_{x \to a} f(x) = 0$$

Now, as

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = 0$$

Thus, the given function f(x) is continuous at x = 0.

8. $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ **at x = 0**

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to 0^-} f(x) = \frac{e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}} = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}\left(1-e^{-1/h}\right)} = \lim_{h \to 0} \frac{1}{e^{1/h}-1} = \frac{1}{e^{1/0}-1}$$

$$= \frac{1}{e^{\infty}-1} = \frac{1}{0-1} = -1 \qquad [\because e^{\infty} = 0]$$

$$\lim_{x \to 0^+} f(x) = \frac{e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h}\left(1+e^{1/h}\right)} = \lim_{h \to 0} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1 \qquad [e^{-\infty} = 0]$$

$$\lim_{x \to 0} f(x) = 0$$

Now, as

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0} f(x)$$

Thus, f(x) is discontinuous at x = 0.

9. $f(x) = \begin{cases} \dfrac{x^2}{2}, & if\ 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & if\ 1 < x \le 2 \end{cases}$ **at x = 1**

Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to 1^-} f(x) = \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2} = \frac{1}{2}$$

$$\lim_{x \to 1} f(x) = \frac{x^2}{2} = \frac{(1)^2}{2} = \frac{1}{2}$$

$$\lim_{x \to 1^+} f(x) = 2x^2 - 3x + \frac{3}{2} = 2(1)^2 - 3(1) + \frac{3}{2} = 2 - 3 + \frac{3}{2} = \frac{1}{2}$$

Now, as

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = \frac{1}{2}$$

Thus, the given function f(x) is continuous at x = 1.

10. $f(x) = |x| + |x - 1|$ at $x = 1$
Solution:

Checking the right hand and left hand limits for the given function, we have

$$\lim_{x \to 1^-} f(x) = |x| + |x-1| = \lim_{h \to 0} |1-h| + |1-h-1|$$
$$= |1-0| + |1-0-1| = 1 + 0 = 1$$

$$\lim_{x \to 1^+} f(x) = |x| + |x-1|$$
$$= \lim_{h \to 0} |1+h| + |1+h-1| = 1 + 0 = 1$$

$$\lim_{x \to 1} f(x) = |x| + |x-1| = |1| + |1-1| = 1 + 0 = 1$$

Now, as

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x)$$

Thus, f(x) is continuous at x = 1.

Find the value of *k* in each of the Exercises 11 to 14 so that the function *f* is continuous at the indicated point:

11.
$$f(x) = \begin{cases} 3x - 8, & if\ x \le 5 \\ 2k, & if\ x > 5 \end{cases} \quad \text{at } x = 5$$

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\lim_{x \to 5^-} f(x) = 3x - 8$$
$$= \lim_{h \to 0} 3(5-h) - 8 = 15 - 8 = 7$$

$$\lim_{x \to 5^+} f(x) = 2k$$

As the function is continuous at $x = 5$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x)$$

So,

7 = 2k

k = 7/2 = 3.5

Therefore, the value of k is 3.5

12. $f(x)=\begin{cases} \dfrac{2^{x+2}-16}{4^x-16}, & if\ x\neq 2 \\ k, & if\ x=2 \end{cases}$ at $x=2$

Solution:

The given function f(x) can be rewritten as,

$$f(x) = \frac{2^{x+2}-16}{4^x-16} = \frac{2^2.2^x-16}{(2^x)^2-(4)^2} = \frac{4(2^x-4)}{(2^x-4)(2^x+4)}$$

$$f(x) = \frac{4}{2^x+4}$$

$$\lim_{x\to 2} f(x) = \lim_{h\to 0} \frac{4}{2^{2-h}+4} = \frac{4}{2^2+4} = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

$$\lim_{x\to 2} f(x) = k$$

As the function is continuous at $x = 2$.

$$\therefore \lim_{x\to 2} f(x) = \lim_{x\to 2} f(x)$$

So, k = ½

Therefore, the value of k is ½

13. $f(x)=\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if\ -1\le x<0 \\ \dfrac{2x+1}{x-1}, & if\ 0\le x\le 1 \end{cases}$ at $x = 0$

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\lim_{x\to 0^-} f(x) = \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$$

$$= \lim_{x\to 0^-} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x} \times \frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$= \lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{1+kx-1+kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x \to 0^-} \frac{2k}{\sqrt{1+kx} + \sqrt{1-kx}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}}$$

$$= \frac{2k}{\sqrt{1}+\sqrt{1}} = \frac{2k}{2} = k$$

$$\lim_{x \to 0^+} f(x) = \frac{2x+1}{x-1} = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

As the function is continuous at $x = 0$.

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$k = -1$$

Therefore, the value of k is -1

14. $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & if\ x \neq 0 \\ \dfrac{1}{2} & , if\ x = 0 \end{cases}$ at $x = 0$

Solution:

Finding the left hand and right hand limits for the given function, we have

$$\lim_{x \to 0^-} f(x) = \frac{1-\cos kx}{x \sin x}$$

$$= \lim_{h \to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)} = \lim_{h \to 0} \frac{1-\cos(-kh)}{-h.\sin(-h)}$$

$$= \lim_{h \to 0} \frac{1-\cos kh}{h \sin h} \qquad \begin{bmatrix} \because \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{bmatrix}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{\substack{h \to 0 \\ kh \to 0}} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} \times \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} . \frac{1}{h.\frac{\sin h}{h}.h}$$

$$= 2.1.\frac{kh}{2}.1.\frac{kh}{2}.\frac{1}{h^2}.1 \qquad \begin{bmatrix} \lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and} \\ \lim_{kh \to 0} \frac{\sin kh}{kh} = 1 \end{bmatrix}$$

$$= \frac{k^2}{2}$$

$$\lim_{x \to 0^+} f(x) = \frac{1}{2}$$

But, as the function is continuous we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\text{So, } \frac{k^2}{2} = \frac{1}{2}$$

$$k^2 = 1 \Rightarrow k = \pm 1$$

Therefore, the value of k is ± 1

15. Prove that the function *f* defined by

$$f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k \quad , & x = 0 \end{cases}$$

remains discontinuous at *x* = 0, regardless the choice of *k*.
Solution:

Finding the left hand and right hand limit for the given function, we have

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \frac{x}{|x| + 2x^2} = \lim_{h \to 0} \frac{0 - h}{|0 - h| + 2(0 - h)^2} \\ &= \lim_{h \to 0} \frac{-h}{h + 2h^2} = \lim_{h \to 0} \frac{-h}{h(1 + 2h)} \\ &= \lim_{h \to 0} \frac{-1}{1 + 2h} = \frac{-1}{1 + 2(0)} = -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \frac{x}{|x| + 2x^2} = \lim_{h \to 0} \frac{0 + h}{|0 + h| + 2(0 + h)^2} \\ &= \lim_{h \to 0} \frac{h}{h + 2h^2} = \lim_{h \to 0} \frac{h}{h(1 + 2h)} = \frac{1}{1 + 0} = 1 \end{aligned}$$

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Now, as the left hand limit and the right hand limit are not equal and the value of both the limits are a constant.

Hence, regardless the choice of k, the given function remains discontinuous at x = 0.

16. Find the values of *a* and *b* such that the function *f* defined by

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at *x* = 4.
Solution:

Finding the left hand and right hand limit for the given function, we have

$$\lim_{x \to 4^-} f(x) = \frac{x-4}{|x-4|} + a$$

$$= \lim_{h \to 0} \frac{4-h-4}{|4-h-4|} + a = \lim_{h \to 0} \frac{-h}{h} + a = -1 + a$$

$$\lim_{x \to 4} f(x) = a + b$$

$$\lim_{x \to 4^+} f(x) = \frac{x-4}{|x-4|} + b$$

$$= \lim_{h \to 0} \frac{4+h-4}{|4+h-4|} + b = \lim_{h \to 0} \frac{h}{h} + b = 1 + b$$

As the function is continuous at $x = 4$.

$$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4} f(x) = \lim_{x \to 4^+} f(x)$$

So, $-1 + a = a + b = 1 + b$

$-1 + a = a + b$ and $1 + b = a + b$

We get, $b = -1$ and $1 + -1 = a + -1 \Rightarrow a = 1$

Therefore, the value of a = 1 and b = -1

17. Given the function *f* (*x*) = 1/(*x* + 2) . Find the points of discontinuity of the composite function *y* = *f* (*f* (*x*)).
Solution:

Given,

$$f(x) = \frac{1}{x+2}$$

$$f[f(x)] = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x+2}+2} = \frac{1}{\frac{1+2x+4}{x+2}} = \frac{x+2}{2x+5}$$

$$\therefore \quad f[f(x)] = \frac{x+2}{2x+5}$$

Now, the function will not be defined and continuous where

$2x + 5 = 0 \Rightarrow x = -5/2$

Therefore, x = -5/2 is the point of discontinuity.

18. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2+t-2}$, where $t = \frac{1}{x-1}$.
Solution:

$$\text{Given, } f(t) = \frac{1}{t^2 + t - 2}$$

$$\Rightarrow f(t) = \frac{1}{\frac{1}{(x-1)^2} + \frac{1}{(x-1)} - 2} \qquad \left[\text{putting } t = \frac{1}{x-1}\right]$$

$$= \frac{1}{\frac{1 + x - 1 - 2(x-1)^2}{(x-1)^2}} = \frac{(x-1)^2}{x - 2x^2 - 2 + 4x}$$

$$= \frac{(x-1)^2}{-2x^2 + 5x - 2} = \frac{(x-1)^2}{-(2x^2 - 5x + 2)}$$

$$= \frac{(x-1)^2}{-[2x^2 - 4x - x + 2]} = \frac{(x-1)^2}{-[2x(x-2) - 1(x-2)]}$$

$$= \frac{(x-1)^2}{-(x-2)(2x-1)} = \frac{(x-1)^2}{(2-x)(2x-1)}$$

Now,
if f(t) is discontinuous, then $2 - x = 0 \Rightarrow x = 2$
And, $2x - 1 = 0 \Rightarrow x = ½$
Therefore, the required points of discontinuity for the given function are 2 and ½.

19. Show that the function *f* (*x*) = |sin *x* + cos *x*| is continuous at *x* = p. Examine the differentiability of *f*, where *f* is defined by
Solution:

Given,
f(x) = |sin x + cos x| at x = π
Now, put g(x) = sin x + cos x and h(x) = |x|
Hence, h[g(x)] = h(sin x + cos x) = |sin x + cos x|
Now,
g(x) = sin x + cos x is a continuous function since sin x and cos x are two continuous functions at x = π.
We know that, every modulus function is a common function is a continuous function everywhere.
Therefore, f(x) = |sin x + cos x| is continuous function at x = π.

20. $f(x) = \begin{cases} x[x], & if\ 0 \le x < 2 \\ (x-1)x, & if\ 2 \le x < 3 \end{cases}$ **at x = 2.**
Solution:

We know that, a function f is differentiable at a point ‘a’ in its domain if
Lf’(c) = Rf’(c)

where $Lf'(c) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ and

$Rf'(c) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Here, $f(x) = \begin{cases} x[x], \text{ if } 0 \le x < 2 \\ (x-1)x, \text{ if } 2 \le x < 2 \end{cases}$ at $x = 2$.
at $x = 2$

$Lf'(c) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$

$= \lim_{h \to 0} \frac{(2-h).1 - 2}{-h}$ $\quad [\because [2-h] = 1]$

$= \lim_{h \to 0} \frac{2-h-2}{-h} = 1$

$Rf'(c) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1).2}{h}$

$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h} = \lim_{h \to 0} \frac{2+h+2h+h^2-2}{h}$

$= \lim_{h \to 0} \frac{3h+h^2}{h} = \lim_{h \to 0} \frac{h(3+h)}{h} = 3$

$Lf'(2) \ne Rf'(2)$

Therefore, f(x) is not differentiable at $x = 2$.

21. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , if \quad x \ne 0 \\ 0 & , if \quad x = 0 \end{cases}$

at $x = 0$.

Solution:

Given,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, \text{ if } x \ne 0 \\ 0, \quad \text{if } x = 0 \end{cases} \text{ at } x = 0$$

For differentiability we know that:

$Lf'(c) = Rf'(c)$

$\therefore Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$

$$= \lim_{h\to 0} \frac{(0-h)^2 \sin \frac{1}{(0-h)} - 0}{-h} = \frac{h^2 . \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= h.\sin\left(\frac{1}{h}\right) = 0 \times \left[-1 \le \sin\left(\frac{1}{h}\right) \le 1\right]$$

$$= 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right) - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h.\sin\left(\frac{1}{h}\right)$$

$$= 0 \times \left[-1 \le \sin\left(\frac{1}{h}\right) \le 1\right] = 0$$

Hence, $Lf'(0) = Rf'(0) = 0$

Therefore, f(x) is differentiable at x = 0

22. $f(x) = \begin{cases} 1+x & , if \ x \le 2 \\ 5-x & , if \ x > 2 \end{cases}$

at $x = 2$.

Solution:

We know that, f(x) is differentiable at x = 2 if

$Lf'(2) = Rf'(2)$

Now,

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{(1+2-h)-(1+2)}{-h} = \lim_{h\to 0} \frac{3-h-3}{-h} = \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[5-(2+h)]-(1+2)}{h} = \lim_{h\to 0} \frac{3-h-3}{h}$$

$$= \frac{-h}{h} = -1$$

So, $Lf'(2) \ne Rf'(2)$

Thus, f(x) is not differentiable at x = 2.

23. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.

Solution:

Given, $f(x) = |x-5|$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-5) \text{ if } x-5<0 \text{ or } x<5 \\ x-5 \text{ if } x-5>0 \text{ or } x>5 \end{cases}$$

For continuity at $x = 5$

$$\text{L.H.L. } \lim_{h\to 5^-} f(x) = -(x-5)$$
$$= \lim_{h\to 0} -(5-h-5) = \lim_{h\to 0} h = 0$$

$$\text{R.H.L. } \lim_{x\to 5^+} f(x) = x-5$$
$$= \lim_{h\to 0} (5+h-5) = \lim_{h\to 0} h = 0$$

L.H.L. = R.H.L.

So, $f(x)$ is continuous at $x = 5$.

Now, for differentiability

$$Lf'(5) = \lim_{h\to 0} \frac{f(5-h)-f(5)}{-h}$$
$$= \lim_{h\to 0} \frac{-(5-h-5)-(5-5)}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$
$$Rf'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0} \frac{(5+h-5)-(5-5)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

Thus, $\quad Lf'(5) \neq Rf'(5)$

Therefore, f(x) is not differentiable at x = 5.

24. A function f : R → R satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y \in$R, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Solution:

Given,

f : R → R satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y \in$R, $f(x) \neq 0$

Let us take any point x = 0 at which the function f(x) is differentiable.

$$\text{So, } f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$2 = \lim_{h\to 0} \frac{f(0).f(h)-f(0)}{h} \quad [\because f(0) = f(h)] \quad ...(i)$$

$$\Rightarrow \quad 2 = \lim_{h\to 0} \frac{f(0)[f(h)-1]}{h}$$

$$\text{Now, } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x).f(h)-f(x)}{h} \quad [\because f(x+y) = f(x).f(y)]$$

$$= \lim_{h \to 0} \frac{f(x)[f(h)-1]}{h} = 2f(x) \qquad \text{from eqn. } (i)$$

Therefore, $f'(x) = 2f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43) :

25. $2^{\cos^2 x}$

Solution:

Let $y = 2^{\cos^2 x}$

Taking log on both sides, we get

$$\log y = \log 2^{\cos^2 x} \Rightarrow \log y = \cos^2 x \,.\, \log 2$$

Now,

Differentiating both sides w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = \log 2.\frac{d}{dx}\cos^2 x$$

$$\frac{1}{y}.\frac{dy}{dx} = \log 2\left[2\cos x.\frac{d}{dx}\cos x\right]$$

$$\frac{1}{y}.\frac{dy}{dx} = \log 2\ [2\cos x\,(-\sin x)]$$

$$\frac{1}{y}.\frac{dy}{dx} = \log 2\ (-\sin 2x)$$

$$\frac{dy}{dx} = -y\,.\log 2 \sin 2x$$

Thus, $\dfrac{dy}{dx} = -2^{\cos^2 x}(\log 2 \sin 2x)$

26. $\dfrac{8^x}{x^8}$

Solution:

Let $y = \dfrac{8^x}{x^8}$

Taking log on both sides, we get $\log y = \log \dfrac{8^x}{x^8}$

$$\Rightarrow \log y = \log 8^x - \log x^8 \Rightarrow \log y = x \log 8 - 8 \log x$$

Differentiating both sides w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = \log 8.1 - \frac{8}{x} \Rightarrow \frac{dy}{dx} = y\left[\log 8 - \frac{8}{x}\right]$$

Thus, $\dfrac{dy}{dx} = \dfrac{8^x}{x^8}\left[\log 8 - \dfrac{8}{x}\right]$

27. $\log\left(x+\sqrt{x^2+a}\right)$

Solution:

Let $y = \log\left(x+\sqrt{x^2+a}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log\left(x+\sqrt{x^2+a}\right)$$

$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+a}\right)$$

$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\left[1+\frac{1}{2\sqrt{x^2+a}}\times\frac{d}{dx}(x^2+a)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\left[1+\frac{1}{2\sqrt{x^2+a}}.2x\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\left[1+\frac{x}{\sqrt{x^2+a}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}\cdot\left(\frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}\right) = \frac{1}{\sqrt{x^2+a}}$$

Thus, $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+a}}$.

28. $\log\left[\log\left(\log x^5\right)\right]$

Solution:

Let, $y = \log[\log(\log x^5)]$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log[\log(\log x^5)]$$

$$= \frac{1}{\log(\log x^5)}\times\frac{d}{dx}\log(\log x^5)$$

$$= \frac{1}{\log(\log x^5)}\times\frac{1}{\log(x^5)}\times\frac{d}{dx}\log x^5$$

$$= \frac{1}{\log(\log x^5)}\cdot\frac{1}{\log(x^5)}\cdot\frac{1}{x^5}\cdot\frac{d}{dx}x^5$$

$$= \frac{1}{\log(\log x^5)} \cdot \frac{1}{\log(x^5)} \cdot \frac{1}{x^5} \cdot 5x^4$$

$$= \frac{5}{x \log(x^5).\log(\log x^5)}$$

Thus, $\frac{dy}{dx} = \frac{5}{x \log(x^5).\log(\log x^5)}$.

29. $\sin\sqrt{x} + \cos^2\sqrt{x}$

Solution:

Let $y = \sin\sqrt{x} + \cos^2\sqrt{x}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\sqrt{x}\right) + \frac{d}{dx}\left(\cos^2\sqrt{x}\right)$$

$$= \cos\sqrt{x}.\frac{d}{dx}\left(\sqrt{x}\right) + 2\cos\sqrt{x}.\frac{d}{dx}\left(\cos\sqrt{x}\right)$$

$$= \cos\sqrt{x}.\frac{1}{2\sqrt{x}} + 2\cos\sqrt{x}\left(-\sin\sqrt{x}\right).\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2\sqrt{x}}.\cos\sqrt{x} - 2\cos\sqrt{x}.\sin\sqrt{x}.\frac{1}{2\sqrt{x}}$$

$$= \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$$

Thus, $\frac{dy}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$.

30. $\sin^n(ax^2 + bx + c)$

Solution:

Let $y = \sin^n(ax^2 + bx + c)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^n(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).\frac{d}{dx}(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).(2ax + b)$$

Thus, $\frac{dy}{dx} = n(2ax + b).\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c)$

31. $\cos\left(\tan\sqrt{x+1}\right)$

Solution:

Let $y = \cos\left(\tan\sqrt{x+1}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\frac{d}{dx}\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{d}{dx}\sqrt{x+1}$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{1}{2\sqrt{x+1}}.1$$

Thus, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x+1}}\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}$

32. $\sin x^2 + \sin^2 x + \sin^2 (x^2)$

Solution:

Let $y = \sin x^2 + \sin^2 x + \sin^2(x^2)$

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) + \frac{d}{dx}\sin^2 x + \frac{d}{dx}\sin^2(x^2)$$

$$= \cos x^2.\frac{d}{dx}(x^2) + 2\sin x.\frac{d}{dx}(\sin x) + 2\sin(x^2)\frac{d}{dx}\sin(x^2)$$

$$= \cos x^2.2x + 2\sin x.\cos x + 2\sin x^2.\cos x^2.\frac{d}{dx}(x^2)$$

$$= 2x.\cos x^2 + \sin 2x + 2\sin x^2.\cos x^2.2x$$

Thus, $\frac{dy}{dx} = 2x.\cos x^2 + \sin 2x + 2x\sin 2x^2$

33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Solution:

Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}}.\frac{d}{dx}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{x+1}}}.\frac{d}{dx}(x+1)^{-1/2}$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}}.\frac{-1}{2}(x+1)^{-3/2}.\frac{d}{dx}(x+1)$$

$$= \frac{\sqrt{x+1}}{\sqrt{x}}.\frac{-1}{2}(x+1)^{-3/2}.1$$

$$= \frac{-1}{2}.\frac{\sqrt{x+1}}{\sqrt{x}}.\frac{1}{(x+1)^{3/2}} = -\frac{1}{2\sqrt{x}(x+1)}$$

Thus, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}(x+1)}$

34. (sin x)$^{\cos x}$

Solution:

Let $y = (\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log y = \cos x.\log(\sin x) \qquad [\because \log x^y = y \log x]$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}\cos x.\log(\sin x)$$

$$\frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{d}{dx}\log(\sin x) + \log(\sin x).\frac{d}{dx}\cos x$$

$$\frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x) + \log(\sin x).(-\sin x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \cot x.\cos x - \sin x.\log(\sin x)$$

$$\frac{dy}{dx} = y[\cot x.\cos x - \sin x.\log(\sin x)]$$

Thus, $\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$

35. $\sin^m x \cdot \cos^n x$
Solution:

Let $y = \sin^m x \cdot \cos^n x$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^m x.\cos^n x)$$

$$= \sin^m x.\frac{d}{dx}(\cos^n x) + \cos^n x.\frac{d}{dx}\sin^m x$$

$$= \sin^m x.n.\cos^{n-1} x\frac{d}{dx}(\cos x) + \cos^n x.m.\sin^{m-1} x$$

$$\frac{d}{dx}(\sin x)$$

$$= n.\sin^m x.\cos^{n-1} x.(-\sin x) + m.\cos^n x.\sin^{m-1} x.\cos x$$

$$= -n.\sin^{m+1} x.\cos^{n-1} x + m\cos^{n+1} x.\sin^{m-1} x$$

$$= \sin^m x.\cos^n x\left[-n\frac{\sin x}{\cos x} + m.\frac{\cos x}{\sin x}\right]$$

Thus, $\frac{dy}{dx} = \sin^m x.\cos^n x\left[-n\tan x + m.\cot x\right]$

36. $(x + 1)^2 + (x + 2)^3 + (x + 3)^4$
Solution:

Let $y = (x+1)^2(x+2)^3(x+3)^4$

Taking log on both sides,

$$\log y = \log[(x+1)^2.(x+2)^3.(x+3)^4]$$

$$\log y = \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

$[\because \log xy = \log x + \log y]$

$\Rightarrow \log y = 2\log(x+1) + 3\log(x+2) + 4\log(x+3)$

$[\because \log x^y = y\log x]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = 2.\frac{d}{dx}\log(x+1) + 3.\frac{d}{dx}\log(x+2) + 4.\frac{d}{dx}\log(x+3)$$

$$\frac{1}{y}.\frac{dy}{dx} = 2.\frac{1}{x+1} + 3.\frac{1}{x+2} + 4.\frac{1}{x+3}$$

$$\frac{dy}{dx} = y\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

Now,

$$\Rightarrow \quad \frac{dy}{dx} = (x+1)^2(x+2)^3(x+3)^4\left[\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right]$$

$$= (x+1)^2(x+2)^3(x+3)^4$$

$$\left[\frac{2(x+2)(x+3)+3(x+1)(x+3)+4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= (x+1)(x+2)^2(x+3)^3(2x^2+10x+12+3x^2+12x+9$$
$$+4x^2+12x+8)$$

$$= (x+1)(x+2)^2(x+3)^3(9x^2+34x+29)$$

Thus, $\frac{dy}{dx} = (x+1)(x+2)^2(x+3)^3(9x^2+34x+29)$

37. $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{-\pi}{4}<x<\frac{\pi}{4}$

Solution:

$$\text{Let } y = \cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$= \cos^{-1}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]$$

$$= \cos^{-1}\left[\sin\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}.\cos x\right]=\cos^{-1}\left[\cos\left(\frac{\pi}{4}-x\right)\right]$$

$$y = \frac{\pi}{4}-x \qquad \left[\because -\frac{\pi}{4}<x<\frac{\pi}{4}\right]$$

Differentiating both sides w.r.t. x

Thus, $\frac{dy}{dx} = -1$

38. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$

Solution:

$$\text{Let } y = \tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]$$

$$= \tan^{-1}\left[\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right] \qquad \left[\begin{array}{l}\because 1-\cos x = 2\sin^2 x/2\\ \quad 1+\cos x = 2\cos^2 x/2\end{array}\right]$$

$$= \tan^{-1}\left[\frac{\sin x/2}{\cos x/2}\right] = \tan^{-1}\left[\tan\frac{x}{2}\right]$$

Thus, $y = \frac{x}{2}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}.1 = \frac{1}{2}$$

Hence, $\frac{dy}{dx} = \frac{1}{2}$

39. $\tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution:

Let $y = \tan^{-1}(\sec x + \tan x)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\sec x + \tan x)]$$

$$= \frac{1}{1+(\sec x + \tan x)^2}.\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \tan x}.(\sec x \tan x + \sec^2 x)$$

$$= \frac{1}{(1+\tan^2 x)+\sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{\sec^2 x + \sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec x(\sec x + \tan x)}.\sec x(\tan x + \sec x) = \frac{1}{2}$$

Thus, $\frac{dy}{dx} = \frac{1}{2}$

40. $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } \frac{a}{b}\tan x > -1$

Solution:

Let $y = \tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$y = \tan^{-1}\left[\dfrac{\dfrac{a\cos x}{b\cos x} - \dfrac{b\sin x}{b\cos x}}{\dfrac{b\cos x}{b\cos x} + \dfrac{a\sin x}{b\cos x}}\right]$$

$$y = \tan^{-1}\left[\dfrac{\dfrac{a}{b} - \tan x}{1 + \dfrac{a}{b}\tan x}\right]$$

$$y = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$\Rightarrow\ y = \tan^{-1}\frac{a}{b} - x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x) = 0 - 1 = -1$$

Thus, $\dfrac{dy}{dx} = -1.$

41. $\sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right),\ 0 < x < \dfrac{1}{\sqrt{2}}$

Solution:

Let $y = \sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right)$

Put $x = \cos\theta \quad \therefore\ \theta = \cos^{-1}x$

$$\Rightarrow\ y = \sec^{-1}\left(\frac{1}{4\cos^3\theta - 3\cos\theta}\right)$$

$$y = \sec^{-1}\left(\frac{1}{\cos 3\theta}\right) \qquad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$$

$$y = \sec^{-1}(\sec 3\theta) \Rightarrow y = 3\theta$$

$$y = 3\cos^{-1}x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\cos^{-1}x = 3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}$$

Thus, $\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$.

42. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), \frac{-1}{\sqrt{3}}<\frac{x}{a}<\frac{1}{\sqrt{3}}$

Solution:

Let $y = \tan^{-1}\left[\frac{3a^2x-x^3}{a^3-3ax^2}\right]$

Put $x = a\tan\theta \quad \therefore \theta = \tan^{-1}\frac{x}{a}$

$$y = \tan^{-1}\left[\frac{3a^2.a\tan\theta - a^3\tan^3\theta}{a^3-3a.a^2\tan^2\theta}\right]$$

$$y = \tan^{-1}\left[\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right]$$

$$y = \tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$y = \tan^{-1}[\tan 3\theta] \qquad \left[\because \tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$\Rightarrow y = 3\theta \Rightarrow y = 3\tan^{-1}\frac{x}{a}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)$$

$$= 3.\frac{1}{1+\frac{x^2}{a^2}}.\frac{d}{dx}.\left(\frac{x}{a}\right) = 3.\frac{a^2}{a^2+x^2}.\frac{1}{a} = \frac{3a}{a^2+x^2}$$

Thus, $\frac{dy}{dx} = \frac{3a}{a^2+x^2}$.

43. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1<x<1, x\neq 0$

Solution:

Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Putting $x^2 = \cos 2\theta \quad \therefore \theta = \dfrac{1}{2}\cos^{-1} x^2$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$y = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$y = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$y = \tan^{-1}\left[\frac{\dfrac{\cos\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}}\right]$$

$$y = \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$y = \tan^{-1}\left[\frac{\tan\dfrac{\pi}{4}+\tan\theta}{1-\tan\dfrac{\pi}{4}.\tan\theta}\right]$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta \quad \Rightarrow y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1} x^2$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{1}{2}\frac{d}{dx}(\cos^{-1} x^2)$$

$$= 0+\frac{1}{2}\times\frac{-1}{\sqrt{1-x^4}}.\frac{d}{dx}(x^2) = \frac{-1.2x}{2\sqrt{1-x^4}} = -\frac{x}{\sqrt{1-x^4}}$$

Thus, $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{1-x^4}}.$

Find *dy/dx* of each of the functions expressed in parametric form in Exercises from 44 to 48.

44. $x = t + \frac{1}{t},\ y = t - \frac{1}{t}$

Solution:

Given,

x = t + 1/t, y = t – 1/t

Differentiating both the parametric functions w.r.t θ

$$\frac{dx}{dt} = 1 - \frac{1}{t^2},\ \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

Thus, $\frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1}$.

45. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right),\ y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Solution:

Given,

$$x = e^{\theta}\left(\theta + \frac{1}{\theta}\right),\ y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

Differentiating both the parametric functions w.r.t. θ.

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right).e^{\theta}$$

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right) \Rightarrow e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$$= \frac{e^{\theta}(\theta^3 + \theta^2 + \theta - 1)}{\theta^2}$$

$$y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right).(-e^{-\theta})$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta}\right) \Rightarrow e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$$= e^{-\theta}\frac{(-\theta^3 + \theta^2 + \theta + 1)}{\theta^2}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$$

Thus, $\frac{dy}{dx} = e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$.

46. $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

Solution:

Given, $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

Differentiating both the parametric functions w.r.t. θ

$$\frac{dx}{d\theta} = -3\sin\theta - 6\cos^2\theta.\frac{d}{d\theta}(\cos\theta)$$

$$= -3\sin\theta - 6\cos^2\theta\,.\,(-\sin\theta)$$

$$= -3\sin\theta + 6\cos^2\theta\,.\,\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 6\sin^2\theta.\frac{d}{d\theta}(\sin\theta)$$

$$= 3\cos\theta - 6\sin^2\theta\,.\,\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta.\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\theta\,(3-6\sin^2\theta)}{\sin\theta\,(-3+6\cos^2\theta)} = \frac{\cos\theta\left[3-6\,(1-\cos^2\theta)\right]}{\sin\theta\left[-3+6\cos^2\theta\right]}$$

$$= \cot\theta\left(\frac{3-6+6\cos^2\theta}{-3+6\cos^2\theta}\right) = \cot\theta\left(\frac{-3+6\cos^2\theta}{-3+6\cos^2\theta}\right)$$

$$= \cot\theta$$

Thus, $\frac{dy}{dx} = \cot\theta$.

47. $\sin x = \frac{2t}{1+t^2},\quad \tan y = \frac{2t}{1-t^2}$.

Solution:

Given,

$\sin x = 2t/(1 + t^2)$, $\tan y = 2t/(1 - t^2)$

Now, taking $\sin x = \dfrac{2t}{1+t^2}$

Differentiating both sides w.r.t t, we get

$$\cos x.\frac{dx}{dt} = \frac{(1+t^2).\frac{d}{dt}(2t) - 2t.\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\cos x.\frac{dx}{dt} = \frac{2(1+t^2) - 2t.2t}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2} \times \frac{1}{\cos x}$$

$$\frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\sin^2 x}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{\frac{(1+t^2)^2-4t^2}{(1+t^2)^2}}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1+t^2}{\sqrt{1+t^4+2t^2-4t^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)}{\sqrt{1+t^4-2t^2}}$$

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{\sqrt{(1-t^2)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{(1-t^2)} \Rightarrow \frac{dx}{dt} = \frac{2}{1+t^2}$$

Now taking, $\tan y = \dfrac{2}{1-t^2}$

Differentiating both sides w.r.t, t, we get

$$\frac{d}{dt}(\tan y) = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y\frac{dy}{dt} = \frac{(1-t^2).\frac{d}{dt}(2t) - 2t.\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\sec^2 y\frac{dy}{dt} = \frac{(1-t^2).2 - 2t.(-2t)}{(1-t^2)^2}$$

$$\sec^2 y \frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2+2t^2}{(1-t^2)^2} \times \frac{1}{\sec^2 y}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\tan^2 y}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{\frac{(1-t^2)^2+4t^2}{(1-t^2)^2}}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^2+2t^2+4t^2}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^4+2t^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{(1+t^2)^2} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\therefore \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$$

Thus, $\frac{dy}{dt} = 1$

48. $x = \frac{1+\log t}{t^2}, \quad y = \frac{3+2\log t}{t}.$

Solution:

On differentiating both the given parametric functions w.r.t. t, we have

$$\frac{dx}{dt} = \frac{t^2 \cdot \frac{d}{dt}(1+\log t) - (1+\log t) \cdot \frac{d}{dt}(t^2)}{t^4}$$

$$= \frac{t^2 \cdot \left(\frac{1}{t}\right) - (1+\log t) \cdot 2t}{t^4} = \frac{t-(1+\log t) \cdot 2t}{t^4}$$

$$= \frac{t[1-2-2\log t]}{t^4} = \frac{-(1+2\log t)}{t^3}$$

$$y = \frac{3+2\log t}{t}$$

Next,

$$\frac{dy}{dt} = \frac{t.\frac{d}{dt}(3+2\log t)-(3+2\log t).\frac{d}{dt}(t)}{t^2}$$

$$= \frac{t(2/t)-(3+2\log t).1}{t^2}$$

$$= \frac{2-3-2\log t}{t^2} = \frac{-(1+2\log t)}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-(1+2\log t)}{t^2}}{\frac{-(1+2\log t)}{t^3}} = \frac{t^3}{t^2} = t$$

Thus, $\frac{dy}{dx} = t.$

49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $dy/dx = -y\log x/x\log y$.
Solution:

Given,
$x = e^{\cos 2t}$ and $y = e^{\sin 2t}$
So, cos 2t = log x and sin 2t = log y
Now, differentiating both the parameter functions w.r.t t, we have

$$\frac{dx}{dt} = e^{\cos 2t}.\frac{d}{dt}(\cos 2t) = e^{\cos 2t}(-\sin 2t).\frac{d}{dt}(2t)$$

$$= -e^{\cos 2t}.\sin 2t.2 = -2e^{\cos 2t}.\sin 2t$$

Now, $y = e^{\sin 2t}$

$$\frac{dy}{dt} = e^{\sin 2t}.\frac{d}{dt}(\sin 2t) = e^{\sin 2t}.\cos 2t.\frac{d}{dt}(2t)$$

$$= e^{\sin 2t}.\cos 2t.2 = 2e^{\sin 2t}.\cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t}.\cos 2t}{-2e^{\cos 2t}.\sin 2t} = \frac{e^{\sin 2t}.\cos 2t}{-e^{\cos 2t}.\sin 2t} = \frac{y\cos 2t}{-x\sin 2t}$$

$$= \frac{y\log x}{-x\log y} \quad \left[\begin{array}{l}\because \cos 2t = \log x \\ \sin 2t = \log y\end{array}\right]$$

Thus, $\frac{dy}{dx} = -\frac{y\log x}{x\log y}.$

50. If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$
Solution:

Given,
$x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$
Differentiating both the parametric equations w.r.t t, we have

$$\frac{dx}{dt} = a\left[\sin 2t.\frac{d}{dt}(1+\cos 2t)+(1+\cos 2t).\frac{d}{dt}\sin 2t\right]$$
$$= a[\sin 2t.(-\sin 2t).2+(1+\cos 2t)(\cos 2t).2]$$
$$= a[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$
$$= a[2(\cos^2 2t - \sin^2 2t) + 2\cos 2t]$$
$$= a[2\cos 4t + 2\cos 2t] \quad [\because \cos 2x = \cos^2 x - \sin^2 x]$$
$$= 2a[\cos 4t + \cos 2t]$$

Now, $y = b\cos 2t(1-\cos 2t)$

$$\frac{dy}{dt} = b\left[\cos 2t.\frac{d}{dt}(1-\cos 2t)+(1-\cos 2t).\frac{d}{dt}(\cos 2t)\right]$$
$$= b[\cos 2t.\sin 2t.2+(1-\cos 2t).(-\sin 2t).2]$$
$$= b[2\sin 2t.\cos 2t - 2\sin 2t + 2\sin 2t\cos 2t]$$
$$= b[\sin 4t - 2\sin 2t + \sin 4t] \quad [\because \sin 2x = 2\sin x\cos x]$$
$$= b[2\sin 4t - 2\sin 2t] = 2b(\sin 4t - \sin 2t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b[\sin 4t - \sin 2t]}{2a[\cos 4t + \cos 2t]} = \frac{b}{a}\left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t}\right]$$

Putting, $t = \frac{\pi}{4}$ we get

$$\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{\sin 4\left(\frac{\pi}{4}\right)-\sin 2.\left(\frac{\pi}{4}\right)}{\cos 4\left(\frac{\pi}{4}\right)+\cos 2.\left(\frac{\pi}{4}\right)}\right] = \frac{b}{a}\left[\frac{\sin\pi - \sin\frac{\pi}{2}}{\cos\pi + \cos\frac{\pi}{2}}\right]$$
$$= \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}\left(\frac{-1}{-1}\right) = \frac{b}{a}$$

Thus, $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}.$

51. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Solution:

Given,

$x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$

Now, differentiating both the parametric functions w.r.t t, we have

$$\frac{dx}{dt} = 3\cos t - \cos 3t.3 = 3(\cos t - \cos 3t)$$
$$\frac{dy}{dt} = -3\sin t + \sin 3t.3 = 3(-\sin t + \sin 3t)$$

So, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3(-\sin t + \sin 3t)}{3(\cos t - \cos 3t)} = \dfrac{-\sin t + \sin 3t}{\cos t - \cos 3t}$

Putting, $t = \dfrac{\pi}{3}$

$$\frac{dy}{dx} = \frac{-\sin\frac{\pi}{3} + \sin 3\left(\frac{\pi}{3}\right)}{\cos\frac{\pi}{3} - \cos 3\left(\frac{\pi}{3}\right)}$$

$$= \frac{-\frac{\sqrt{3}}{2} + \sin\pi}{\frac{1}{2} - \cos\pi} = \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} - (-1)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2} + 1} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{-1}{\sqrt{3}}$$

Thus, $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{3}}$.

52. Differentiate x/sinx w.r.t sin x.
Solution:

Let $y = \dfrac{x}{\sin x}$ and $z = \sin x$.

Differentiating both the parametric functions w.r.t. x,

$$\frac{dy}{dx} = \frac{\sin x.\frac{d}{dx}(x) - x.\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x.1 - x.\cos x}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$\frac{dz}{dx} = \cos x$$

Now, $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{\frac{\sin x - x\cos x}{\sin^2 x}}{\cos x} = \dfrac{\sin x - x\cos x}{\sin^2 x\cos x}$

$$= \frac{\sin x}{\sin^2 x\cos x} - \frac{x\cos x}{\sin^2 x\cos x}$$

$$= \frac{\tan x}{\sin^2 x} - \frac{x}{\sin^2 x} = \frac{\tan x - x}{\sin^2 x}$$

Thus, $\dfrac{dy}{dz} = \dfrac{\tan x - x}{\sin^2 x}$.

53. Different $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ **w.r.t $\tan^{-1} x$ when $x \neq 0$.**
Solution:

Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1} x$

Now, put $x = \tan\theta$

$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$ and $z = \tan^{-1}(\tan\theta) = \theta.$

So, $\tan\left(\frac{\sqrt{\sec\ \theta}-1}{\tan}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$$\tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right) = \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$\Rightarrow y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$

Differentiating both parametric functions w.r.t. θ

$\frac{dy}{d\theta} = \frac{1}{2}.\frac{d}{d\theta}(\theta)$ and $\frac{dz}{d\theta} = \frac{d}{d\theta}(\theta)$

$= \frac{1}{2}.1 = \frac{1}{2}$ and $\frac{dz}{d\theta} = 1$

Thus, $\frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = \frac{1/2}{1} = \frac{1}{2}.$

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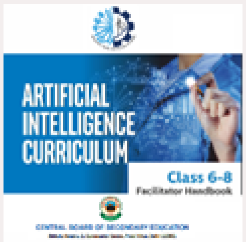

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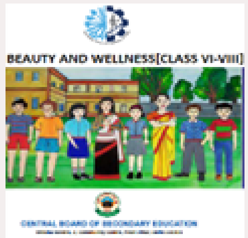

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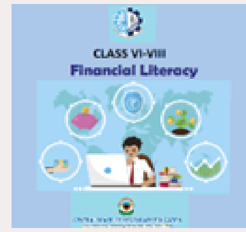

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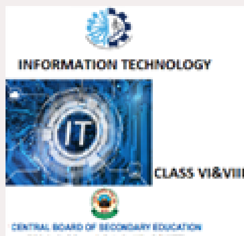
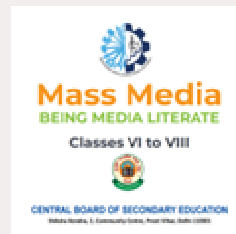
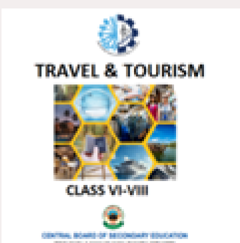


Information Technology

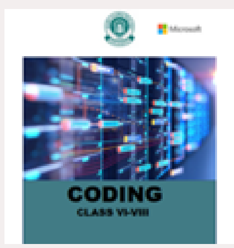
Marketing/Commercial Application



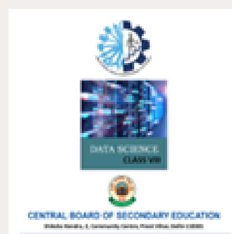
Mass Media - Being Media Literate



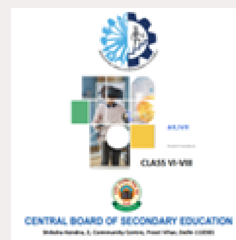
Travel & Tourism



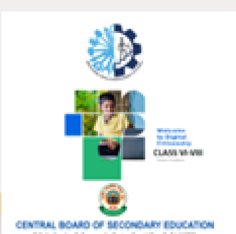
Coding



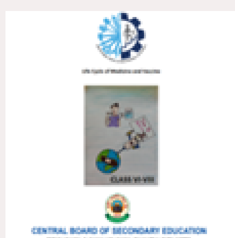
Data Science (Class VIII only)



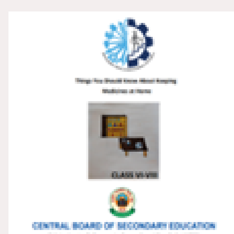
Augmented Reality / Virtual Reality



Digital Citizenship

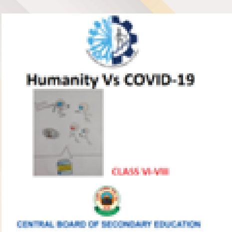


Life Cycle of Medicine & Vaccine

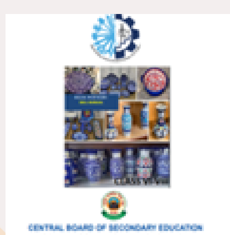


Things you should know about keeping Medicines at home

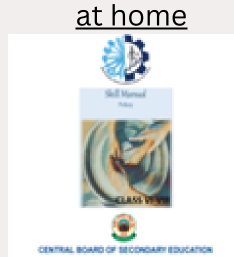
What to do when Doctor is not around



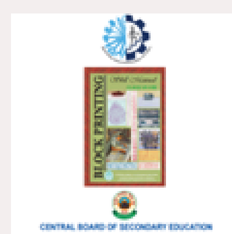
Humanity & Covid-19



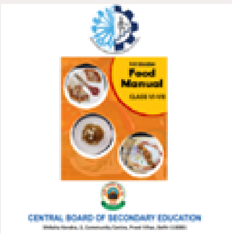
Blue Pottery



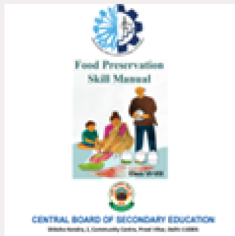
Pottery



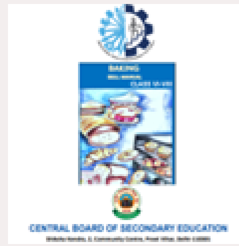
Block Printing



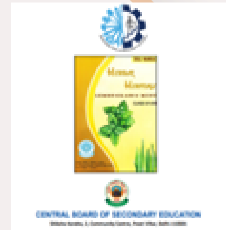
Food



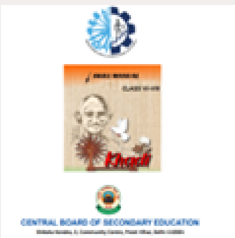
Food Preservation



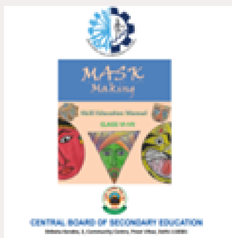
Baking



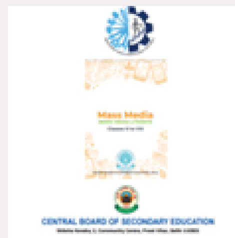
Herbal Heritage



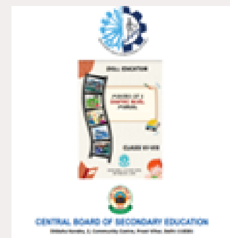
Khadi



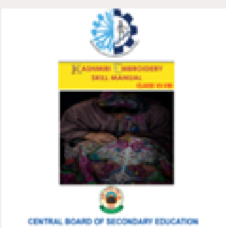
Mask Making



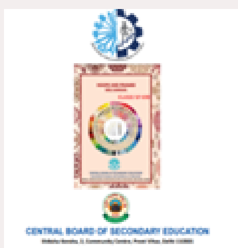
Mass Media



Making of a Graphic Novel

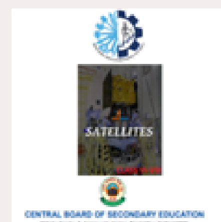


Kashmiri Embroidery

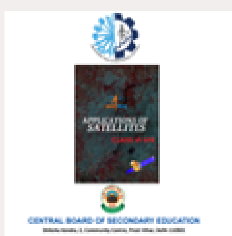


Embroidery

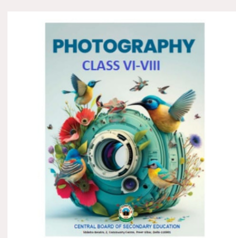
Rockets



Satellites

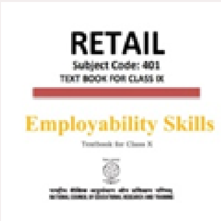


Application of Satellites



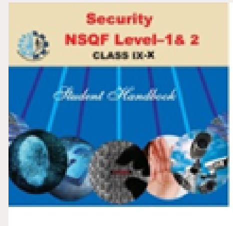
Photography

SKILL SUBJECTS AT SECONDARY LEVEL (CLASSES IX – X)

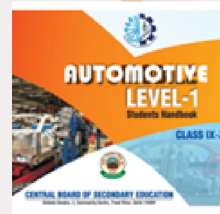


Retail

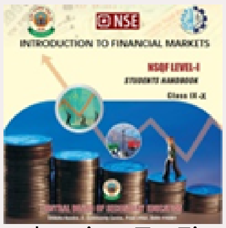
Information Technology



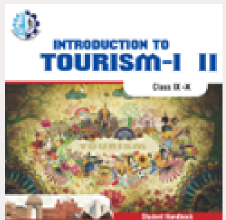
Security



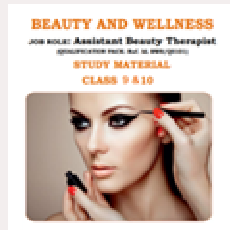
Automotive



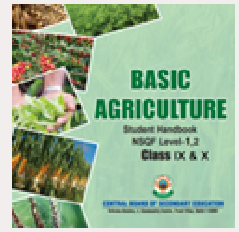
Introduction To Financial Markets



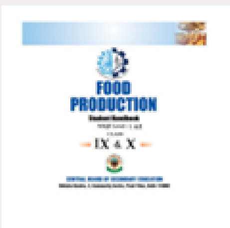
Introduction To Tourism



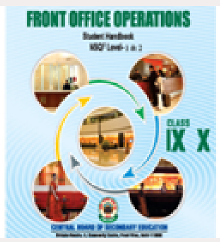
Beauty & Wellness



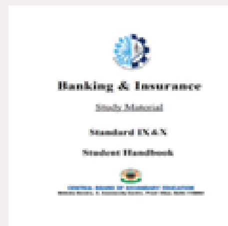
Agriculture



Food Production

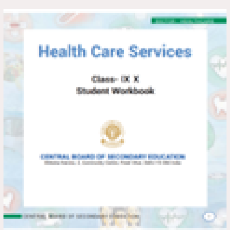


Front Office Operations



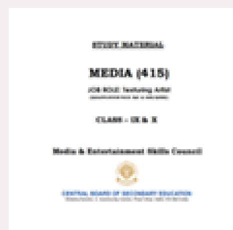
Banking & Insurance

Marketing & Sales

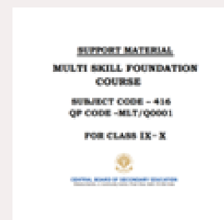


Health Care

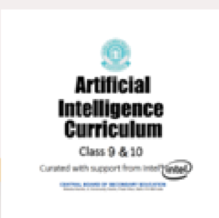
Apparel



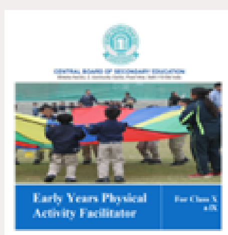
Multi Media



Multi Skill Foundation Course



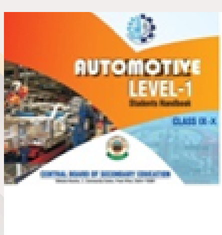
Artificial Intelligence



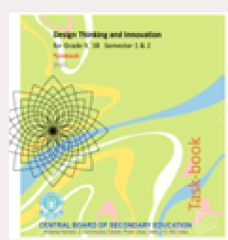
Physical Activity Trainer

Data Science

Electronics & Hardware (NEW)

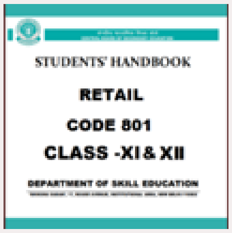


Foundation Skills For Sciences (Pharmaceutical & Biotechnology)(NEW)

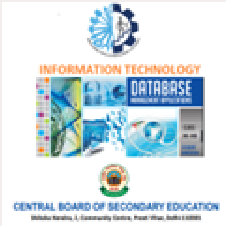


Design Thinking & Innovation (NEW)

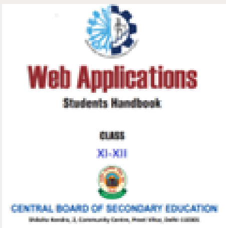
SKILL SUBJECTS AT SR. SEC. LEVEL (CLASSES XI – XII)



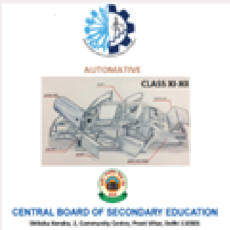
Retail



InformationTechnology

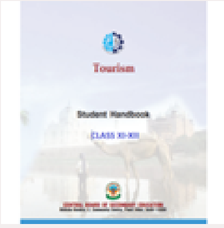


Web Application

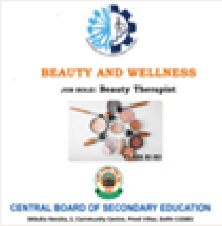


Automotive

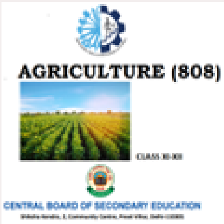
Financial Markets Management



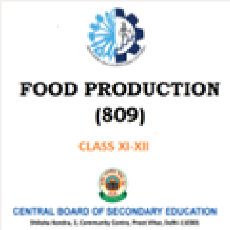
Tourism



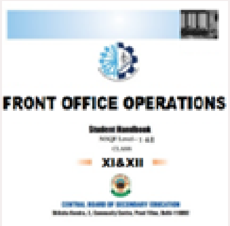
Beauty & Wellness



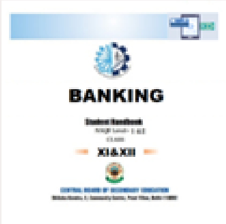
Agriculture



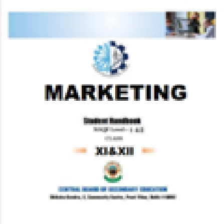
Food Production



Front Office Operations



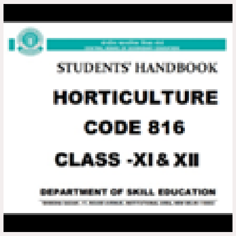
Banking



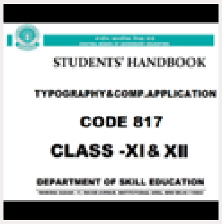
Marketing

Health Care

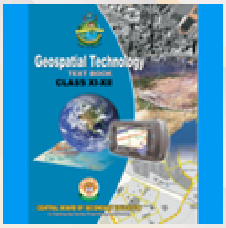
Insurance



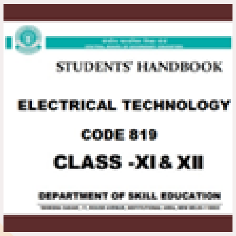
Horticulture



Typography & Comp. Application

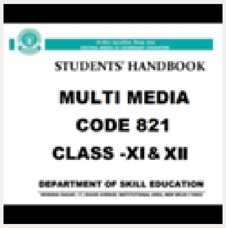


Geospatial Technology

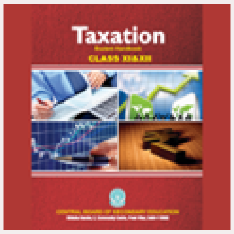


Electrical Technology

Electronic Technology

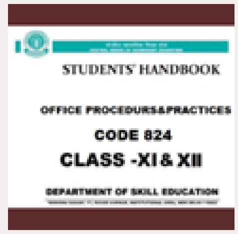


Multi-Media

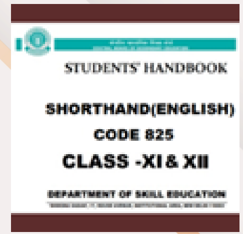


Taxation

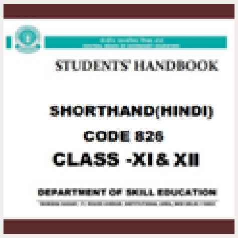
Cost Accounting



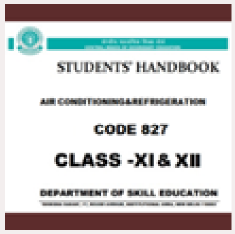
Office Procedures & Practices



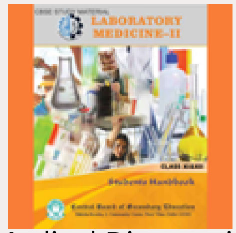
Shorthand (English)



Shorthand (Hindi)

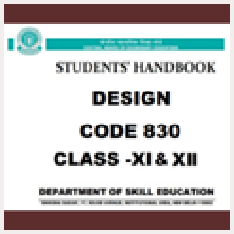


Air-Conditioning & Refrigeration

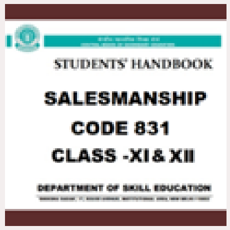


Medical Diagnostics

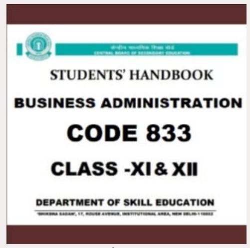
Textile Design



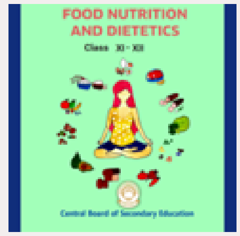
Design



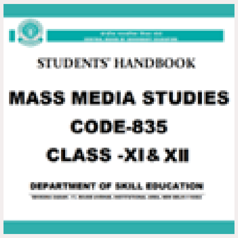
Salesmanship



Business Administration

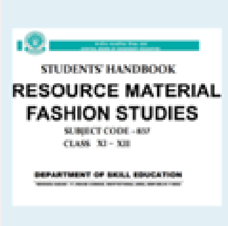


Food Nutrition & Dietetics

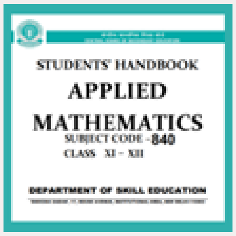


Mass Media Studies

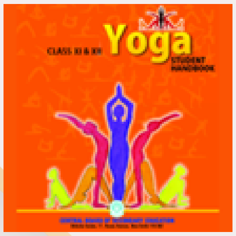
Library & Information Science



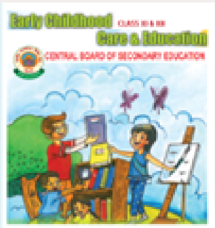
Fashion Studies



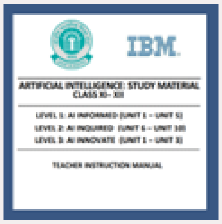
Applied Mathematics



Yoga

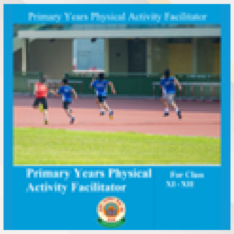


Early Childhood Care & Education

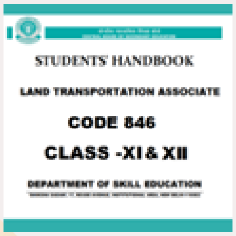


Artificial Intelligence

Data Science



Physical Activity Trainer(new)



Land Transportation Associate (NEW)

Electronics & Hardware (NEW)

Design Thinking & Innovation (NEW)